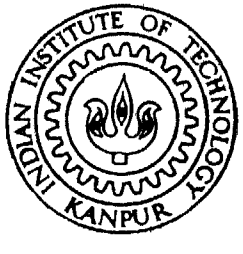
EVALUATION OF CONVOLUTION INTEGRAL USING WAVELET TRANSFORM A STUDY

by

DINESH PRATAP SINGH

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DEPARTMENT OF ELECTRICAL ENGINEERING

INDIAN INSTITUTE OF TECHNOLOGY KANPUR

July, 1996

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for the Degree of

DIIT

by

Dinesh Pratap Singh

to the

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INDIAN INSTITUTE OF TECHNOLOGY, KANPUR
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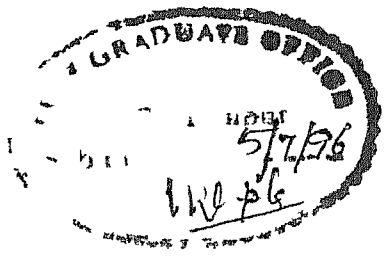
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Certificate



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July 1996

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Abstract

In this thesis the evaluation of convolution integral using continuous wavelet transform has been studied for two time domain signals The results of the convolution product thus obtained has been compared with that of the direct convolution integral method The results indicate that the choice of the wavelet basis function and the corresponding dilation and translation parameters play an important role in the correct evaluation of the convolution integral

Dedicated To

My Parents

Acknowledgements

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July 1996 **Dinesh Pratap Singh**

Contents

1 Introduction 1

1 1 Organisation of the thesis 3

2 Wavelet Theory 4

2 1 Wavelet Transform 5

2 1 1 Continuous Wavelet Transform 5

2 1 2 Discrete Parameter Wavelet Transform 8

2 1 3 Discrete Time Wavelet Transform 10

2 1 4 Discrete Wavelet Transform 10

2 2 Multiresolution Analysis 11

3 Convolution And Wavelet Transform 15

3 1 Convolution Integral 15

3 2 Convolution Theorem 15

3 3 Convolution Using Continuous Wavelet Transform 16

3 4 Convolution Using DPWT 18

4 Implementation And Results 21

4 1 Choice of Wavelet Basis 21

4 2 Implementation of Convolution Integral 23

4 2 1 Ramp and Gate Signals 24

4 2 2 Exponential Signals 25

4 3 Results And Discussion 31

4 4 Conclusion And Future Work 34

List of Figures

2 1	Wavelet Transform Structure	7
2 2	Inverse Wavelet Transform Structure	9
2 3	MRA Decomposition Scheme	12
2 4	MRA Reconstruction Scheme	13
4 1	Second Derivative of Gaussian Wavelet	22
4 2	F T of Second Derivative of Gaussian Wavelet	22
4 3	Direct Convolution of The Ramp and The Gate Signal	26
4 4	Convolution of Ramp and Gate Signal Using Wavelet Transform	26
4 5	Convolution of Ramp and Gate Signal Using Wavelet Transform	27
4 6	Convolution of Ramp and Gate Signal Using Wavelet Transform	27
4 7	Convolution of Ramp and Gate Signal Using Wavelet Transform	28
4 8	Convolution of Ramp and Gate Signal Using Wavelet Transform	28
4 9	Convolution of Ramp and Gate Signal Using Wavelet Transform	29
4 10	Convolution of Ramp and Gate Signal Using Wavelet Transform	30
4 11	Convolution of Ramp and Gate Signal Using Wavelet Transform	30
4 12	Direct Convolution of Two Exponentials ($\alpha = \beta = 1$)	35
4 13	Direct Convolution of Two Exponentials with Varying $\alpha(\beta = 1)$	35
4 14	Convolution of Two Exponentials with Different $\alpha(\beta = 1)$ Using Wavelet Transform	36
4 15	Convolution of Two Exponentials with Differnet $\alpha(\beta = 1)$ Using Wavelet Transform	36

4 16 Convolution of Two Exponentials with Different $\alpha(\beta = 1)$ Using Wavelet Transform 37

4 17 Convolution of Two Exponentials with Differnet $\alpha(\beta = 1)$ Using Wavelet Transform 37

Chapter 1

Introduction

In recent times wavelet analysis (also called wavelet theory or just wavelet) has attracted much attention in signal processing It has been successfully applied in many applications such as transient signal analysis, image analysis communication systems, and other signal processing applications Its attraction is due to its ability to analyse rapidly changing transient signals It can be shown that every application using the Fourier Transform (or FFT) can be formulated using wavelets [1] to provide more localized temporal (or spatial) and frequency information

In signal processing wavelets are very useful for processing non stationary signals When analysing such signals with Fourier Transform any abrupt change in time in the signal x(t) is spread out over the entire frequency axis in X(f) where X(f) is the Fourier Transform of x(t) Therefore analysis adapted to the non stationary signals requires methods other than the Fourier Transform in order to know the frequency content of the signal with change in time that is we need a two dimensional time frequency representation of the signal x(t) Short Time Frequency Transform(STFT) is one such representation which uses time window function of constant width But this has some drawbacks as far as the frequency and time resolution is considered [1]

In the STFT analysis of a non stationary signal there are two conflicting requirements the window width must be long enough to give the desired frequency resolution but must also be short enough so as not to blur the time dependent events Given a window function g(t) and its Fourier transform G(f) the frequency resolution Δ_f and

the time resolution Δ_t are defined as [2]

$$\Delta_f^2 = \frac{\int f^2 |G(f)|^2 df}{\int |G(f)|^2 df} \tag{1 1}$$

and

$$\Delta_t^2 = \frac{\int t^2 |g(t)|^2 dt}{\int |g(t)|^2 dt} \tag{1 2}$$

The frequency resolution and the time resolution are related by the uncertainty principle as

$$\Delta_t \Delta_f \geq \frac{1}{4\pi} \tag{1 3}$$

i e their product is lower bounded Equation1 3 indicates that one can only trade frequency resolution for time resolution or *vice versa* More important is that once a window has been chosen for the STFT then the time-frequency resolution given by equation 1 1 and equation 1 2 is fixed over the entire time frequency plane

The Wavelet Transform representation for non stationary signals overcomes this limitation through the use of varying window widths by choosing different scales for baby wavelets [3] Let $\psi(t)$ be the mother wavelet which has $\Delta_f = \delta_f$ and $\Delta_t = \delta_t$ then $\psi(t/a)$ baby wavelet will have $\Delta_f = \delta_f/a$ and $\Delta_t = a\delta_t$ *where* a is the scale parameter By varying a various time and frequency resolutions can be obtained although of course not both simultaneously

The wavelet theory has been applied in various fields In computer vision multiresolution techniques have been used for various problems ranging from motion estimation to object recognition [4] Images are successfully approximated starting from coarse version and going to a fine resolution version

One of the most important applications for wavelets is in signal compression /coding of image and speech Applications of wavelet decomposition in numerical analysis e g for solving partial differential equations seem very promising because of the *zooming* property which allows a very good representation of discontinuities unlike the Fourier Transform [4] Statistical signal processing using wavelets is another application of wavelets

In signal compression the multiresolution technique resolves the input signal $f(t)$ into components of different resolutions A coding scheme can then asign more bits to represent the lower detail sequences which have shorter lengths due to successive subsamplings and less to the higher detail sequences in order to achieve compression

One particular application of continuous wavelet transform is to perform linear filtering As we will see in chapter 3 it is possible to obtain convolution of two signals with the continuous wavelet transform The complexity of this algorithm is proportional to N If however we achieve the convolution with the FFT algorithm the complexity of the algorithm is $Nlog_2N$ the computing time is thus longer than the one obtained with the Wavelet Transform if we concentrate the energy in very few coefficients

In the present thesis work the computation of convolution using continuous wavelet transform has been undertaken and the results have been discussed

1 1 Organisation of the thesis

The thesis has been presented in four chapters In chapter 2 the wavelet theory in general and different wavelet transforms alongwith multiresolution analysis has been discussed Chapter 3 presents the theory and the algorithm for computing convolution using continuous wavelet transform The results and the conclusions are given in chapter 4

Chapter 2

Wavelet Theory

The mathematics associated with building a model for a signal system or process with a set of *special signals'* or "*wavelets* as known popularly is called as the **wavelet theory** The wavelets must be oscillatory and have amplitudes which quickly decay to zero in both positive and negative directions Wavelet theory represents *things* by breaking them down into many interrelated pieces when the pieces are scaled and translated wavelets this breaking down process is termed as the *wavelet decomposition* or *wavelet transform* *Wavelet reconstruction* or *inverse wavelet transform* involves putting the wavelet pieces back together to retrieve the original object or process Wavelet theory involves the scaling or warping operation and the translation or shifting operation If the translation operation is combined with the scaling action then the composite operation is referred to as an *affine operation* [5] The affine operation simultaneously scales and translates the independent variable Mathematically the affine operation maps the original independent variable t into a new independent variable t by the formula

$$t' = \frac{t - b}{a} \tag{2 1}$$

where a is the scale and b is the translation

2 1 Wavelet Transform

Like Fourier analysis, wavelet analysis deals with the expansion of functions in terms of a set of basis functions but unlike Fourier analysis wavelet analysis expands functions not in terms of the trigonometric polynomials but in the form of translations and dilations of mother wavelet The Fourier transform of a given signal gives the frequency content of the signal However it fails to provide information about the time evolution of the frequency content i e , how the frequency contents of the given signal change with time This time frequency localization is achieved by the *Wavelet Transform*(WT) Thus for the analysis of *non−stationaary siganl* the wavelet transform provides an alternative to the classical Short Time Fourier Transform (STFT) or Gabor transform In contrast to the STFT which uses a single analysis window the WT uses short windows at high frequencies and long windows at low frequencies, so that the time resolution becomes arbitrarily good at high frequencies while the frequency resolution becomes arbitrarily good at low frequencies For example two very close short bursts can always be eventually seperated in the analysis by going up to higher analysis frequencies in order to increase time resolution This kind of analysis of course woiks best if the signal is composed of high frequency components of short duration plus low frequency components of long duration which is often the case with signals encountered in practice There are four different types of wavelet transform [3]

2 1 1 Continuous Wavelet Transform

The *Continuous Wavelet Transform* (CWT) of a signal $f(t)$ maps the one dimensional function into a two dimensional function of scale a and translation b and is defined as,

$$CWT(a\ b) = \frac{1}{\sqrt{a}} \int f(t)\psi \left(\frac{t-b}{a}\right) dt \qquad (2\ 2)$$

where the signal $f(t)$ is assumed to be square integrable denoted as

$$f(t) \in \mathbf{L}^2(\mathbf{R}) \tag{2 3}$$

meaning

$$\int f^2(t)dt < \infty \tag{2 4}$$

and $\psi(t)$ is the *basic* (or *mother*) wavelet ψ (t) denotes the complex conjugate of $\psi(t)$ $\psi(t)$ can be a real or a complex valued function which satisfies the following conditions

$$\int_{-\infty}^{\infty} \mid \psi(t) \mid^2 dt < \infty \tag{2 5}$$

and

$$C_\psi = \int_0^\infty \frac{\mid \Psi(\mathbf{w}) \mid^2}{\mid w \mid} dw < \infty \tag{2 6}$$

Where $\Psi(\mathbf{w})$ is the Fourier transform of $\psi(t)$ The first condition implies finite energy of the function $\psi(t)$ and the second condition is the admissibility condition In the CWT the variable t scale a and shift b are all continuous Fig 2 1 shows the schematic of obtaining the wavelet domain coefficients which are computed for each particular scale and translation $(a\ b)$ value Thus we see that the continuous wavelet transform is an analysis filter which breaks a signal down into component pieces and these pieces are examined or operated instead of the original function

The *inverse continuous wavelet transform* maps a surface in the scale translation plane H(a b) into the one dimensional time domain to reconstruct the original signal $f(t)$ and is defined as

$$f(t) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{a>0}^{\infty} CWT(a,b) \frac{1}{\sqrt{a}} \psi \left(\frac{t-b}{a} \right) \frac{da\ db}{a^2} \tag{2 7}$$

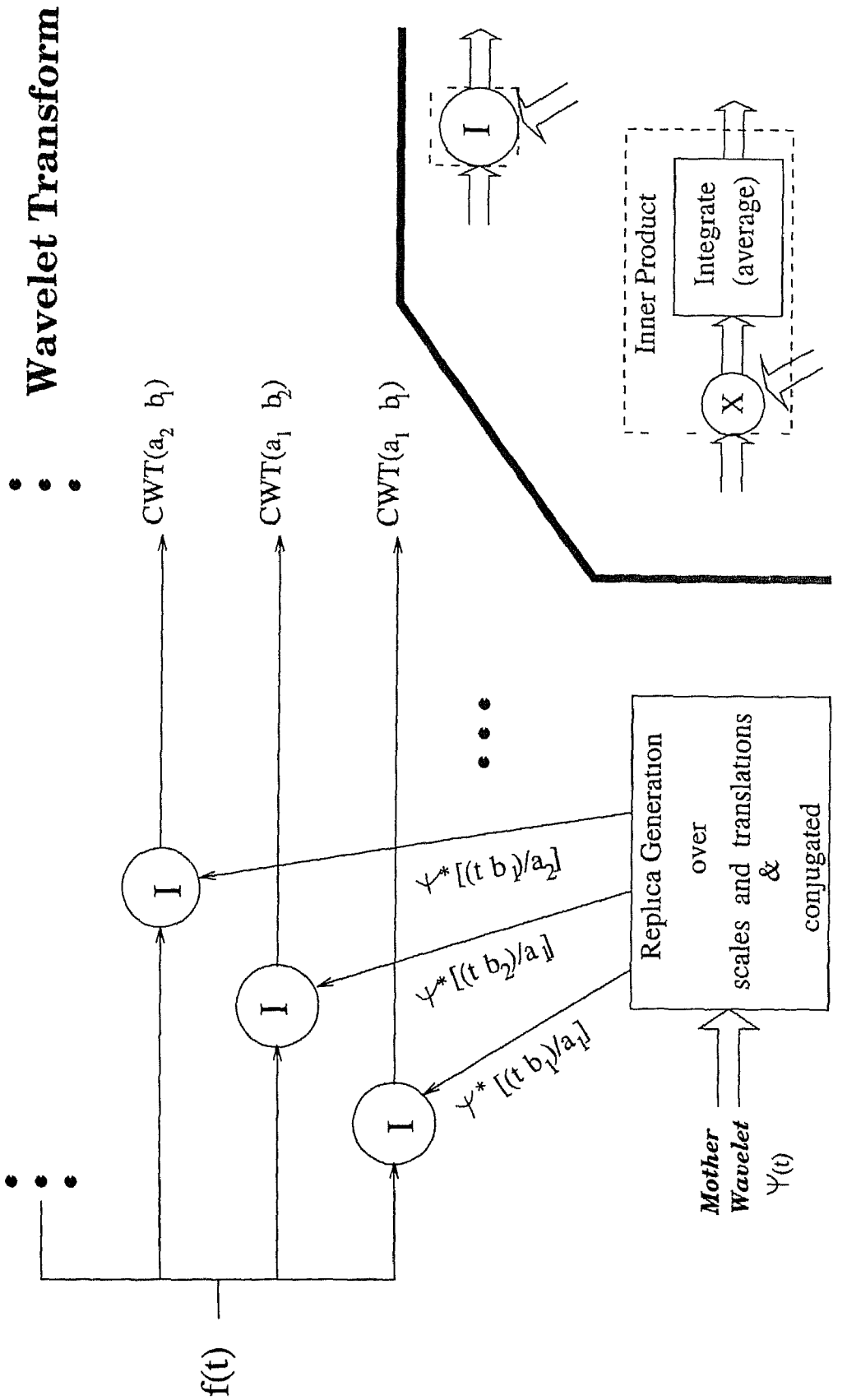


Figure 2 1 Wavelet Transform Structure

The inverse transform thus creates the original function by summing appropriately weighted scaled and translated versions of the mother wavelet The weights are the wavelet coefficients $CWT(a\ b)$ Figure 2 2 illustrates the reconstruction of original signal $f(t)$ from the wavelet domain coefficients The inverse transform or the recon struction filter is a synthesis filter The synthesis filter puts the pieces back together again A characteristic of the inverse transform is that it is non unique in the sense that several different wavelet transform domain representations can be inverted with respect to the same mother wavelet to create the time function $f(t)$ That is the inverse transform may be a many to one operator This non-uniqueness is due to the increase in the dimensionality

2 1 2 Discrete Parameter Wavelet Transform

In the continuous wavelet transform both scale a and translation b are continuous vari ables and there is a redundancy in the CWT representation of $(a\ b)$ The *Discrete Parameter Wavelet Transform*$(DPWT)$ of a continuous signal takes into consider ation this fact and it is defined as [3]

$$DPWT(m\ n) = a_0^{-\frac{m}{}} \int f(t)\psi(a_0^{-m}t - nb_0)dt \tag{2 8}$$

where the parameters a, and b are discretized to $a = a_0^m$ and $b = nb_0a_0^m$ with a_0 b_0 the sampling intervals and m n integers Both $f(t)$ and $\psi(a_0^{-m}t)$ are still continuous For computational efficiency $a_0 = 2$ and $b_0 = 1$ are commonly used resulting in a bin ary dilation of 2^{-n}and a dyadic translation of $2^m n$ Equ ation 2 8 can also be written in the inner product form as

$$DPWT(m\ n) =< f(t)\ \psi_{nn(t)} > \tag{2 9}$$

where $\psi_{mn}(t) = a_0^{-\frac{m}{2}}\psi(a_0^{-m}t - nb_0)$

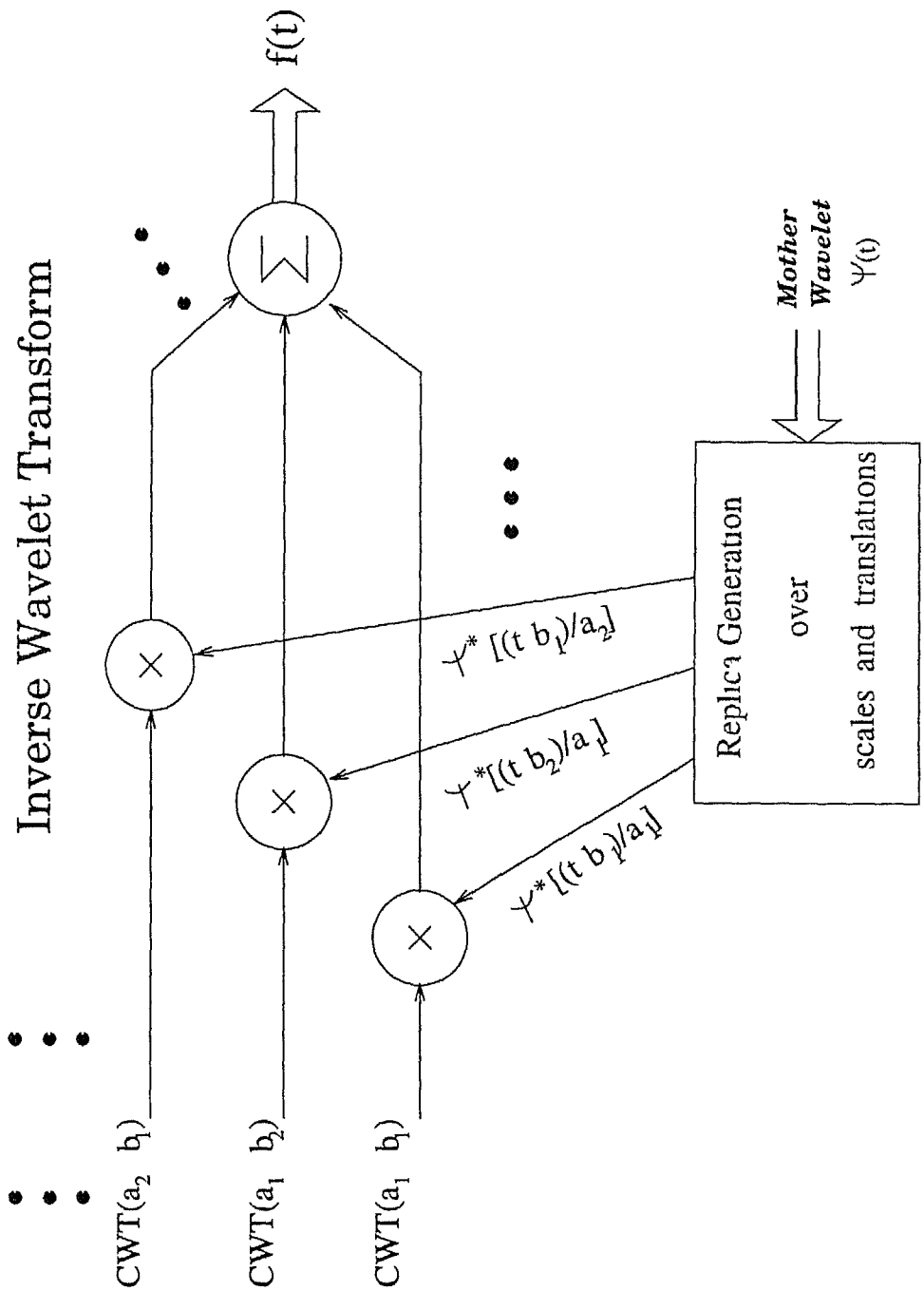


Figure 2 2 Inverse Wavelet Transform Structure

The original time signal can be reconstructed from the $DPWT(m\ n)$ as

$$f(t) = c \sum_m \sum_n DPWT(m\ n)\psi_{mn}(t) \tag{2 10}$$

where c is some constant dependent only on $\psi(t)$ In case of oversampling the above equation will not only hold but moreover it is possible to have a non unique represen tation of f(t) with respect to the same $\psi(t)$ That is different sets of $(a_0\ b_0)$ resulting in different $\psi_{mn}(t)$ will produce different $DPWT(m\ n)$, each capable of recovering the signal $f(t)$ In contrast if the sampling is sparse it may not be possible to recover $f(t)$ at all

2 1 3 Discrete Time Wavelet Transform

The *Discrete Time Wavelet Transform*$(DTWT)$ is a time discretization of $DPWT$ with $t = kT$ and the sampling interval $T = 1$ It is defined as

$$DTWT(m\ n) = a_0^{-\frac{m}{2}} \sum_k f(k)\psi(a_0^{-m}k - nb_0) \tag{2 11}$$

We note that for $a_0 = 2$ there is an output only at every 2^m samples when $2^{-m}k$ is an integer

2 1 4 Discrete Wavelet Transform

The discrete wavelet transform is defined as

$$DWT(m\ n) = 2^{-\frac{m}{2}} \sum_k s(k)\psi(2^{-m}k - n) \tag{2 12}$$

where the *discrete wavelet* $\psi(k)$ can be but not necessarily a sampled version of a continuous counterpart When $\psi(k)$ is a discretization of a $\psi(t)$ the DWT is identical to the $DTWT$

2 2 Multiresolution Analysis

Multiresolution analysis is the decomposition of a signal $f(t)$ into components of different scales (frequency) of 2^{-m} $m \in Z$ [6] Associated with each scale is a subspace V_m These subspaces are time functions which satisfy the following conditions stated below [4]

i Containment

$$\{0\} \rightarrow \quad V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \quad \rightarrow \mathbf{L}^2(\mathbf{R}) \tag{2 13}$$

ii Upward Completeness

$$\bigcup_{m \in Z} V_m = \mathbf{L}^2(\mathbf{R}) \tag{2 14}$$

iii Downward Completeness

$$\bigcap_{m \in Z} V_m = \{0\} \tag{2 15}$$

iv Scale Invariance

$$f(t) \in V_m \Longleftrightarrow f(2^m t) \in V_0 \tag{2 16}$$

v Shift Invariance

$$f(t) \in V_0 \Longleftrightarrow f(t-n) \in V_0 \; for \; all \; n \in Z \tag{2 17}$$

vi Existence of orthonormal scaling functions

There exists a scaling function $\phi(t) \in V_0$ such that the set

$$\left\{\phi_{mn}(t) = 2^{-\frac{m}{2}} \phi(2^{-m}t - n) \quad n \in Z\right\} \tag{2 18}$$

is an orthonormal basis that spans V_m

Figure 2 3 shows the MRA decomposition scheme and figure 2 4 gives the structure for reconstruction The multiresolution analysis scheme decomposes $f(t)$ into the detail components d_n^m $m = 1, 2 \quad L$ and the coarsest approximation c_n^L via stages of identical low pass filters $\overline{g}(l)$ and high pass filters $\overline{h}(l)$ and subsampling the outputs

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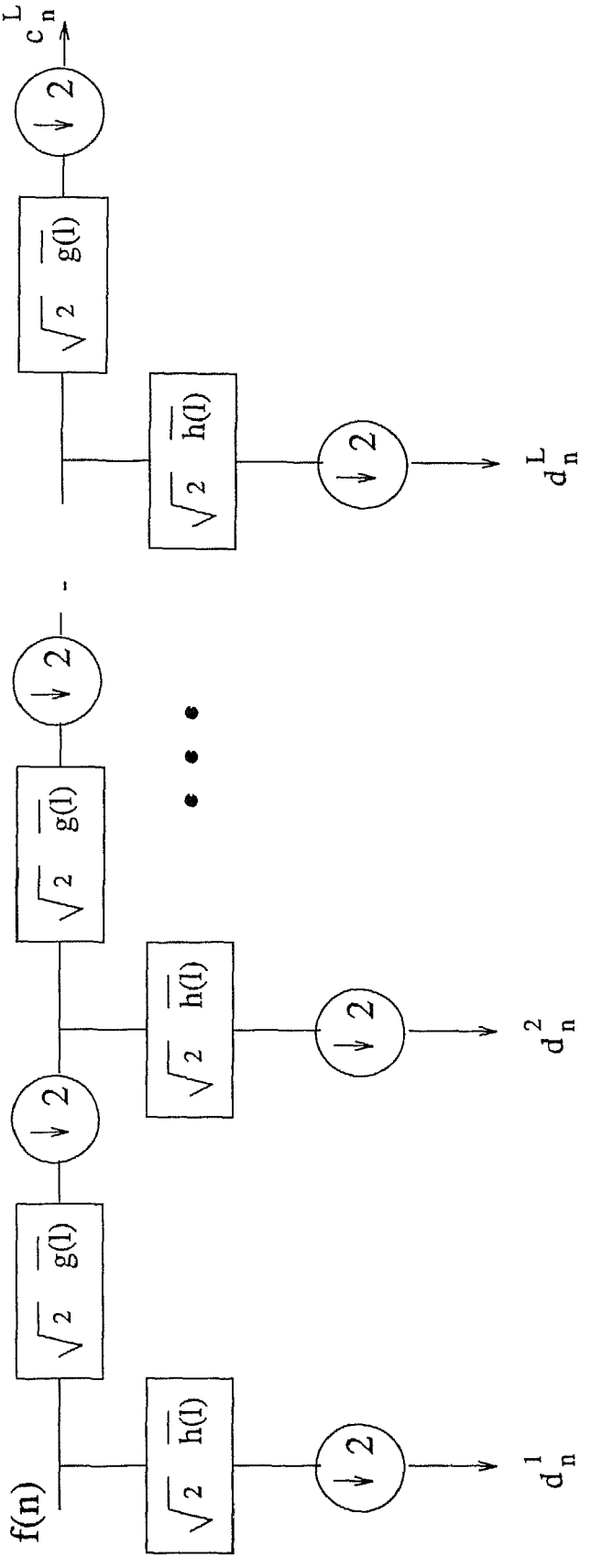


Figure 2 3 MRA Decomposition Scheme

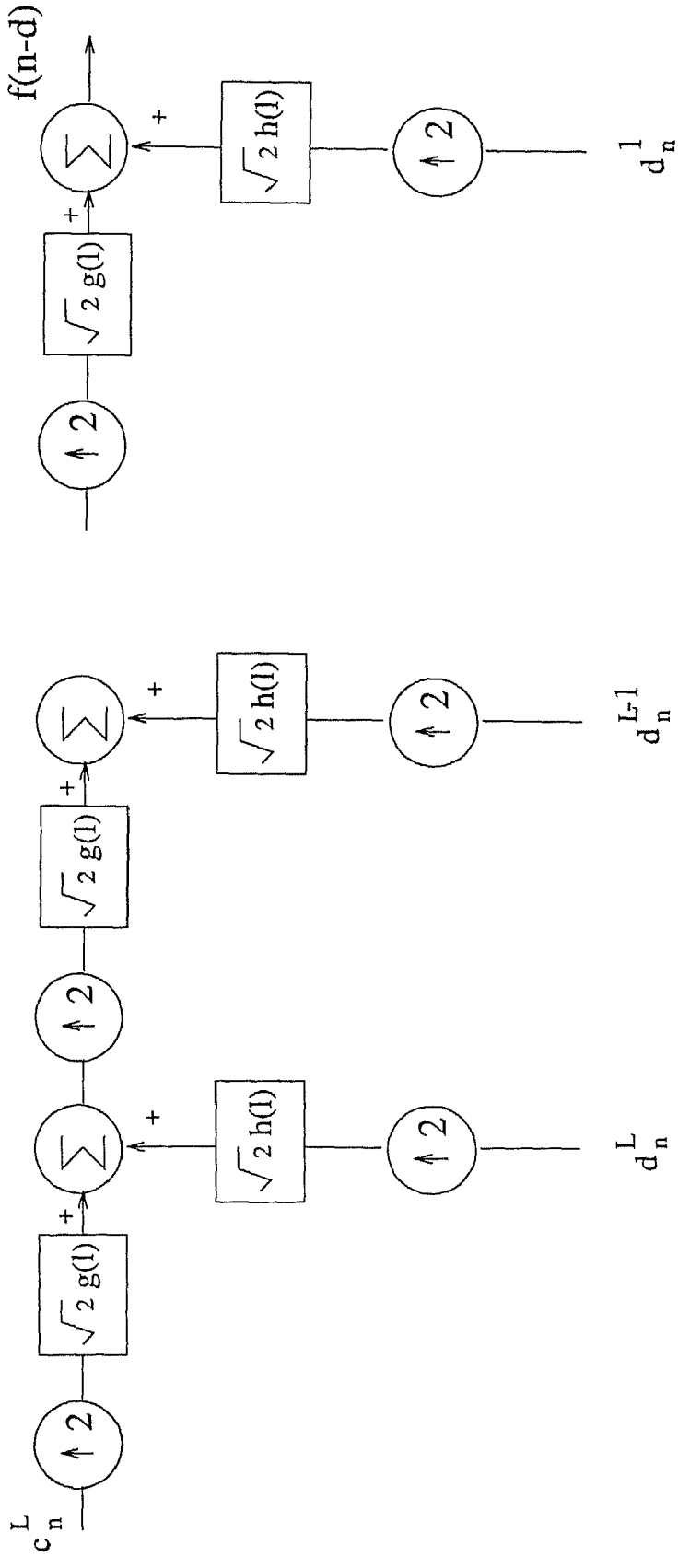


Figure 2 4 MRA Reconstruction Scheme

The decomposition continues until it reaches c_N^L conceivably the dc component of $f(t)$ The subsampling by two produces a time compression by two and identical $\overline{g}(l)(\overline{h}(l))$ will have bandwidths of the lower(upper) half of the previous $\overline{g}(l)(\overline{h}(l))$ The details d_n^m are the DPWT coefficients of $f(t)$ i e $d_n^m = DPWT(m\ n)$ under certain conditions[3] The choices of $g(l)$ and $h(l)$ determine $\phi(t)$ and $\psi(t)$ The filters $g(l)$, $h(l)$ $\overline{g}(l)$ $\overline{h}(l)$ are related to scaling function and wavelet function as [3]

$$\phi(t) = 2\sum_{l=0}^{p-1} g(l)\phi(2t-l) \tag{2 19}$$

$$\psi(t) = 2\sum_{l=0}^{p-1} h(l)\phi(2t-l) \tag{2 20}$$

$$h(l) = (-1)^l g(p-1-l) \tag{2 21}$$

$$\overline{h}(l) = h(p-1-l) \tag{2 22}$$

$$\overline{g}(l) = g(p-1-l) \tag{2 23}$$

where the delay $p-1$ is the filter order The above scheme is the *Mallat algorithm* for *MRA* also known as the *Fast Wavelet Transform* when attached with a WT interpretation

The algorithm has complexity $O(N)$ and is therefore faster than FFT The complexity is independent of the number of stages of decomposition Since a subsampling by two follows each stage the total no of computation is

$$T = T_0 + \frac{T_0}{2} + \frac{T_0}{4} + \qquad = 2T_0 \tag{2 24}$$

where T_0 is the no of computations for the first stage

Chapter 3

Convolution And Wavelet Transform

Amongst several applications of wavelet theory we consider the problem of computing the convolution product of two time signals using wavelet transform

3 1 Convolution Integral

The convolution integral of two functions $f(x)$ and $g(x)$ is defined as

$$h(x) = \int_{-\infty}^{\infty} f(u)g(x-u)du \tag{3 1}$$

and is denoted as

$$h(x) = f(x) * g(x) \tag{3 2}$$

For *Linear Time Invariant* (*LTI*) systems the convolution integral represents the output $h(x)$ of the system in terms of the input signal $f(x)$ and the system impulse response $g(x)$ [7] Convolution integral is the basic time domain formula in evaluating the system response

3 2 Convolution Theorem

If $F_1(w)$ is the Fourier Transform of $f_1(x)$ and $F_2(w)$ is the Fourier Transform of $f_2(x)$ i e

$$f_1(x) \longleftrightarrow F_1(w) \tag{3 3}$$

and

$$f_2(x) \longleftrightarrow F_2(w) \tag{3 4}$$

then

$$y(x) = f_1(x) * f_2(x) \longleftrightarrow F_1(w)F_2(w) \tag{3 5}$$

i e

$$Y(w) = F_1(w)F_2(w) \tag{3 6}$$

where $y(x)$ is the convolution product of $f_1(x)$ and $f_2(x)$ and $Y(w)$ is the Fourier Trans form of $y(x)$ Convolution Theorem is an important theorem applied in evaluating the system response in the frequency domain

3 3 Convolution Using Continuous Wavelet Transform

We will examine here the computation of a convolution using continuous wavelet trans form as presented in [8] Let us consider the convolution product of two functions $f(x)$ and $g(x)$

$$h(x) = \int_{-\infty}^{\infty} f(u)g(x-u)du \tag{3 7}$$

Introducing two real wavelet functions $\psi(x)$ and $\chi(x)$ such that

$$C = \int_0^{\infty} \frac{\widehat{\psi}^*(\nu)\widehat{\chi}(\nu)}{\nu} d\nu \tag{3 8}$$

is defined Here $\widehat{\psi}(\nu)$ and $\widehat{\chi}(\nu)$ denotes the Fourier Transforms of the *wavelets* $\psi(x)$ and $\chi(x)$ respectively

The Wavelet Transform of $g(x)$ with respect to the mother wavelet $\psi(x)$ is

$$W_g(a\ b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} g(x)\psi\ \left(\frac{x-b}{a}\right) dx \tag{3 9}$$

The inverse wavelet transform of $W_g(a\ b)$ with respect to the wavelet function $\chi(x)$ can be written as

$$g(x) = \frac{1}{C}\int_0^\infty \int_{-\infty}^\infty \frac{1}{\sqrt{a}} W_g(a\ b)\chi\left(\frac{x-b}{a}\right)\frac{da\ db}{a^2} \tag{3 10}$$

Substituting $g(x)$ in equation 3 7 we get

$$h(x) = \int_{-\infty}^\infty f(u)\frac{1}{C}\int_0^\infty \int_{-\infty}^\infty \frac{1}{\sqrt{a}} W_g(a,b)\chi\left(\frac{x-u-b}{a}\right)\frac{da\ db}{a^2}du \tag{3 11}$$

$$h(x) = \frac{1}{C}\int_0^\infty \frac{da}{a^{\frac{5}{}}}\int_{-\infty}^\infty W_g(a\ b)db \int_{-\infty}^\infty f(u)\chi\left(\frac{x-u-b}{a}\right)du \tag{3 12}$$

We define

$$\widetilde{\chi}(x) = \chi(-x) \tag{3 13}$$

so that the WT of $f(x)$ with respect to the wavelet function $\widetilde{\chi}(x)$ is

$$\widetilde{W}_f(a\ b) = \frac{1}{\sqrt{a}}\int_{-\infty}^\infty f(x)\widetilde{\chi}\left(\frac{x-b}{a}\right)dx \tag{3 14}$$

Now the integral from equation 3 12 can be written as

$$\begin{aligned}\int_{-\infty}^\infty f(u)\chi\left(\frac{x-u-b}{a}\right)du &= \int_{-\infty}^\infty f(u)\chi\left[-\left\{\frac{u-(x-b)}{a}\right\}\right]du \\ &= \int_{-\infty}^\infty f(u)\widetilde{\chi}\left[\frac{u-(x-b)}{a}\right]du \\ &= \left[\frac{1}{\sqrt{a}}\int_{-\infty}^\infty f(u)\widetilde{\chi}\left[\frac{u-(x-b)}{a}\right]du\right]\sqrt{a}\end{aligned}$$

i e

$$\int_{-\infty}^\infty f(u)\chi\left(\frac{x-u-b}{a}\right)du = \sqrt{a}\,\widetilde{W}_f(a, x-b) \tag{3 15}$$

Using the above expression we may write from equation 3 12

$$h(x) = \frac{1}{C}\int_0^\infty \frac{da}{a^2}\int_{-\infty}^\infty \widetilde{W}_f(a\ x-b)W_g(a,b)db \tag{3 16}$$

but the second integral of equation 3 16 can be represented as the convolution product of $\widetilde{W}_f(a\ x)$ and $W_g(a\ x)$ Then we get the final result

$$h(x) = \frac{1}{C}\int_0^{\infty} \widetilde{W}_f(a\ \imath) \otimes W_g(a\ x)\frac{da}{a^2} \qquad (3\ 17)$$

Therefore in order to compute a convolution with the continuous wavelet transform

- we compute the *wavelet transform* $\widetilde{W}_f(a\ b)$ of the function $f(x)$ with the wavelet function $\tilde{\chi}(x)$
- we compute the *wavelet transform* $W_g(a\ b)$ of the function $g(x)$ with the wavelet function $\psi(x)$
- we compute the convolution product of the wavelet transforms $\widetilde{W}_f(a\ x)$ and $W_g(a, x)$
- we sum this convolution for all scale values from 0 to ∞

The Wavelet Transform permits us to perform any linear filtering Its efficiency depends on the number of terms in the WT associated with $g(x)$ for a given signal $f(x)$ If we have a filter where the number of significant coefficients is small for each scale the complexity of the algorithm is proportional to N For a classical convolution the complexity is also proportional to N but the number of operations is also proportional to the length of the convolution mask The main advantage of the present technique lies in the possibility of having a filter with long scale terms without computing the convolution on a large window If we achieve the convolution with the FFT algorithm the complexity is in $Nlog_2N$ the computing time is longer than the one obtained with the wavelet transform if we concentrate the energy on very few coefficients

3 4 Convolution Using DPWT

In case of *Discrete Parameter Wavelet Transform* ($DPWT$) the wavelet transform is not taken for all scale and shift values but computed only for fixed number of these

values as decided by the sampling grid The convolution of two continuous signals using discrete parameter wavelet transform has been derived and presented below

Considering again two real wavelets $\psi(x)$ and $\chi(x)$, the $DPWT$ of g(x) with respect to the wavelet function $\psi(x)$ is given by

$$W_g(m\ n) = \int_{-\infty}^{\infty} g(x)\psi_{mn}(x) \tag{3 18}$$

where $\psi_{mn}(x) = 2^{-\frac{n}{2}}\psi(2^{-m}x - n)$ for dyadic sampling We can recover g(x) with the wavelet function $\chi(x)$ so that

$$g(x) = c\sum_{m}\sum_{n} W_g(m\ n)\chi_{mn}(x) \tag{3 19}$$

where $\chi_{mn}(x) = 2^{--}\chi(2^{-m}x-n)$ for dyadic sampling and c is some constant dependent only on the wavelet $\chi(x)$

Substituting g(x) from equation 3 19 in equation 3 7 we get,

$$h(x) = c\sum_{m}\sum_{n} W_g(m,n)\int_{-\infty}^{\infty} f(u)\chi_{mn}(x-u)du \tag{3 20}$$

Denoting $\widetilde{\chi}(x) = \chi(-x)$, the $DPWT$ of $f(x)$ with respect to the wavelet function $\widetilde{\chi}(x)$ is

$$\widetilde{W}_f(m\ n) = \int_{-\infty}^{\infty} f(x)\widetilde{\chi}_{mn}(x)dx \tag{3 21}$$

where $\widetilde{\chi}_{mn}(x) = 2^{-\frac{m}{2}}\widetilde{\chi}(2^{-m}x - n)$ Now consider the integral from equation 3 20

$$\int_{-\infty}^{\infty} f(u)\chi_{mn}(x-u)du = \int_{-\infty}^{\infty} f(u)2^{-\frac{m}{2}}\chi_{mn}\left[2^{-m}(x-u) - n\right]du$$

$$= 2^{-\frac{m}{2}}\int_{-\infty}^{\infty} f(u)\chi_{mn}\left[-\left\{n - 2^{-n}(x-u)\right\}\right]du$$

$$= 2^{-\frac{m}{2}}\int_{-\infty}^{\infty} f(u)\widetilde{\chi}\left[n - 2^{-m}(x-u)\right]du$$

$$= 2^{-\frac{m}{2}}\int_{-\infty}^{\infty} f(u)\widetilde{\chi}\left[2^{-m}u - (2^{-m}x - n)\right]du$$

i e

$$\int_{-\infty}^{\infty} f(u)\chi_{mn}(x-u)du = \widetilde{W}_f\left(m\ 2^{-m}x - n\right) \tag{3 22}$$

From equation 3 20 and equation 3 22 we get the final expression for h(x) as

$$h(x) = c\sum_m \sum_n W_g(m,n)\widetilde{W}_f(m\ 2^{-m}x - n) \tag{3 23}$$

The above equation is similar to that obtained with the continuous wavelet transform And we can obtain the convolution product of two signals using the $DPWT$

Chapter 4

Implementation And Results

4 1 Choice of Wavelet Basis

In implementing the computation of convolution through wavelet transform we need two real wavelets $\psi(t)$ and $\chi(t)$ such that $C = \int_0^\infty \frac{\widehat{\psi}(\nu)\widehat{\chi}(\nu)}{\nu} d\nu$ is defined In the present method we have chosen $\psi(t)$ and $\chi(t)$ such that

$$\psi(t) = \chi(t) \tag{4 1}$$

where $\psi(t)$ is an even function of time The wavelet $\psi(t)$ chosen in our implementation is the second derivative of Gaussian wavelet which is an even function of time and it is defined as [3]

$$\psi(t) = (1 - t^2)e^{-\frac{t}{2}} \tag{4 2}$$

and its F T is given by

$$\Psi(w) = \sqrt{2\pi} w^2 e^{-} \tag{4 3}$$

These functions are plotted in the figure 4 1 and figure 4 2

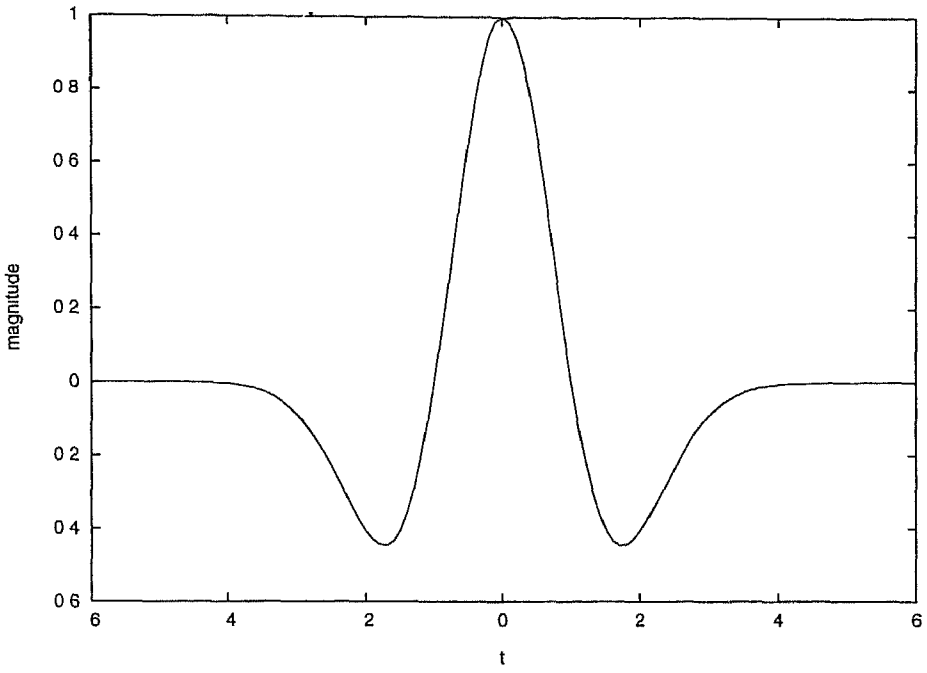


Figure 4 1 Second Derivative of Gaussian Wavelet

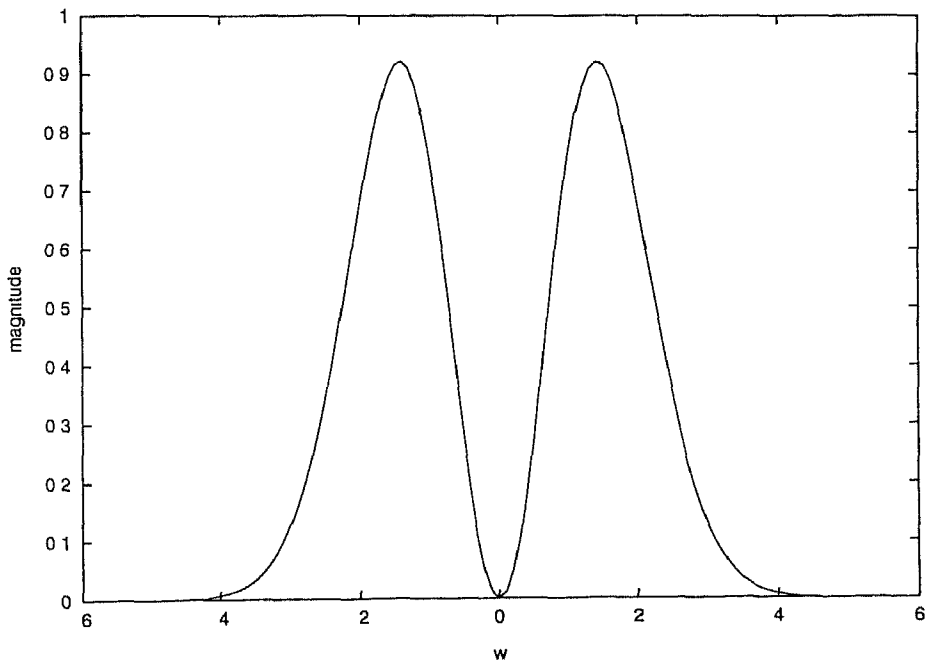


Figure 4 2 F T of Second Derivative of Gaussian Wavelet

4 2 Implementation of Convolution Integral

From equation 3 9, the continuous wavelet transform of the signal g(t) with respect to the wavelet function $\psi(t)$ is given as

$$W_g(a\ b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} g(t)\psi\left(\frac{t-b}{a}\right) dt \tag{4 4}$$

substituting $\psi(t)$ from equation 4 2 in the above expression we may write

$$W_g(a\ b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} g(t)\left[1 - \left(\frac{t-b}{a}\right)^2\right] e^{-\frac{\left(\frac{t-b}{a}\right)^2}{2}} dt \tag{4 5}$$

and similarly the continuous wavelet transform of the signal $f(t)$ with respect to the wavelet function $\psi(t)$ is given as,

$$\widetilde{W}_f(a\ b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(t)\left[1 - \left(\frac{t-b}{a}\right)^2\right] e^{-\frac{\left(\frac{t-b}{a}\right)^2}{2}} dt \tag{4 6}$$

Now changing the time variable $t = u$ in the equation 4 6 we get

$$\widetilde{W}_f(a,b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(u)\left[1 - \left(\frac{u-b}{a}\right)^2\right] e^{-\frac{\left(\frac{-b}{a}\right)^2}{2}} du \tag{4 7}$$

substituting equation 4 5 and equation 4 7 in equation 3 16 we get the expression for $h(x)$ as

$$h(x) = \frac{1}{C}\int_0^{\infty} \frac{da}{a^2} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(t)\left\{1 - \left(\frac{t+b-x}{a}\right)^2\right\} e^{-\frac{\left(\frac{t+b-}{a}\right)^2}{}} dt\right]$$

$$\left[\frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} g(u)\left\{1 - \left(\frac{u-b}{a}\right)^2\right\} e^{-\frac{\left(\frac{-b}{a}\right)^2}{}} du\right] db \tag{4 8}$$

i e

$$h(x) = \frac{1}{C}\int_0^{\infty} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{a^3} f(t)g(u)\left[1 - \left(\frac{t+b-x}{a}\right)^2\right]\left[e^{-\frac{\left(\frac{t+b-}{a}\right)}{2}}\right]$$

$$\left[1 - \left(\frac{u-b}{a}\right)^2\right]\left[e^{-\frac{\left(\frac{-b}{a}\right)^2}{2}}\right] du\ dt\ db\ da \tag{4 9}$$

Equation 4 9 involving four integrals was evaluated using Gaussian Legendre m point formula method [9] Two different sets of input signals $f(t)$ & $g(t)$ were taken The first set was simulated with a ramp signal and a gate signal and the other set with two exponentials having different time constants

4 2 1 Ramp and Gate Signals

The functions $f(t)$ and $g(u)$ were taken to be gate signals and ramp signals respectively as defined below

$$f(t) = 1 \quad 0 \leq t \leq 5 \tag{4 10}$$
$$= 0 \quad elsewhere$$

and

$$g(u) = u \quad 0 \leq u \leq 10 \tag{4 11}$$
$$= 0 \quad elsewhere$$

Therefore in equation 4 9 substituting equation 4 10 and equation 4 11 and changing the limits accordingly we get

$$h(\iota) = \frac{1}{C}\int_0^{\infty}\int_{-\infty}^{\infty}\int_0^5\int_0^{10}\frac{1}{a^3}u\left[1-\left(\frac{t+b-x}{a}\right)^2\right]\left[e^{-\frac{(t+b-)}{}}\right]$$
$$\left[1-\left(\frac{u-b}{a}\right)^2\right]\left[e^{-\frac{(-b)}{}}\right]du\,dt\,db\,da \tag{4 12}$$

We have calculated the convolution integral of a gate signal as given by equation 4 10 and a ramp signal as given by equation 4 11 by using direct integration, using

time domain convolution method graphically Considering the appropriate limits for different overlapping regions of the functions f(u) and g(x u), the convolution product of f(x) and g(x) can be obtained as [7]

$$\begin{aligned} h(x) &= 0 \quad x < 0 \\ &= \tfrac{1}{2}x^2 \quad 0 < x < 5 \\ &= 5x - 12\,5\,, 5 < x < 10 \\ &= -\tfrac{1}{2}x^2 + 5x + 37\,5\,, 10 < x < 15 \\ &= 0 \quad 15 < x \end{aligned} \tag{4 13}$$

The convolution integral so calculated is shown in the figure 4 3

4 2 2 Exponential Signals

Two exponentials having different decay constants were also taken for performing con volution using continuous wavelet transform The functions $f(t)$ and $g(t)$ are defined as

$$f(t) = \exp(-\alpha t)u(t) \tag{4 14}$$

and

$$g(t) = \exp(-\beta t)u(t) \tag{4 15}$$

Therefore in equation 4 9 substituting equation 4 14 and equation 4 15 we get

$$h(\iota) = \frac{1}{C}\int_0^\infty \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{a^3}\exp(-\alpha t)\exp(-\beta u)\left[1-\left(\frac{t+b-x}{a}\right)^2\right]\left[e^{-\frac{\left(\frac{t+b-x}{}\right)}{}}\right]$$
$$\left[1-\left(\frac{u-b}{a}\right)^2\right]\left[e^{-\frac{\left(\frac{-b}{}\right)^2}{2}}\right] du\, dt\, db\, da \tag{4 16}$$

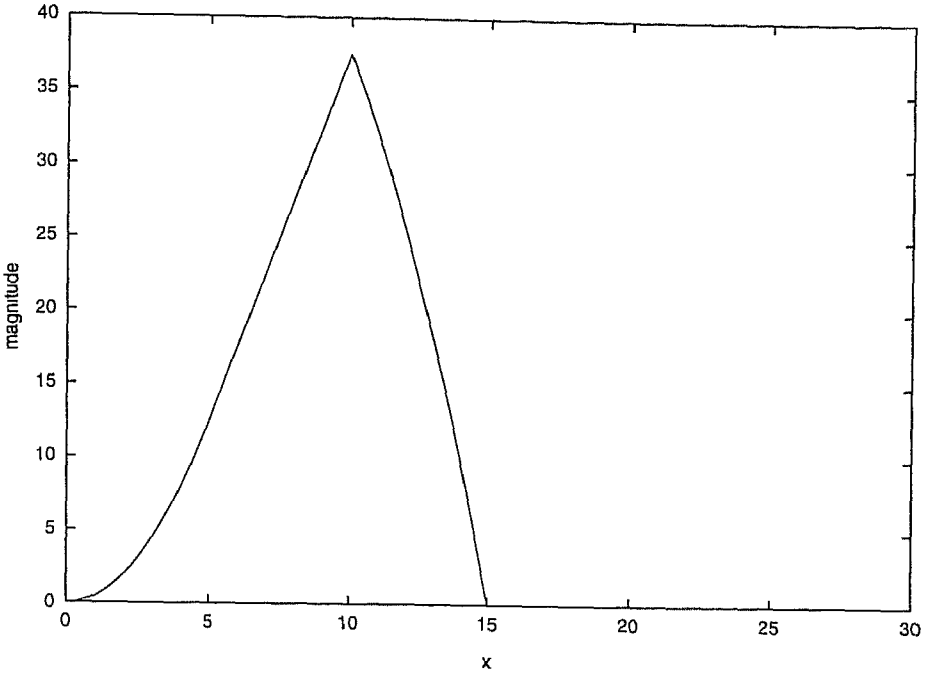


Figure 4 3 Direct Convolution of The Ramp and The Gate Signal

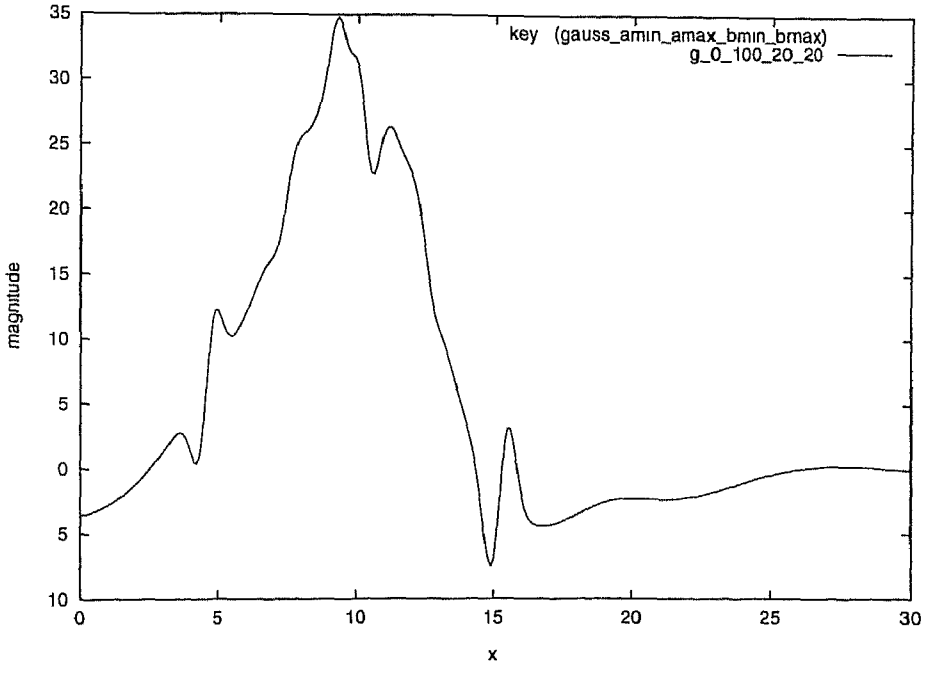


Figure 4 4 Convolution of Ramp and Gate Signal Using Wavelet Transform

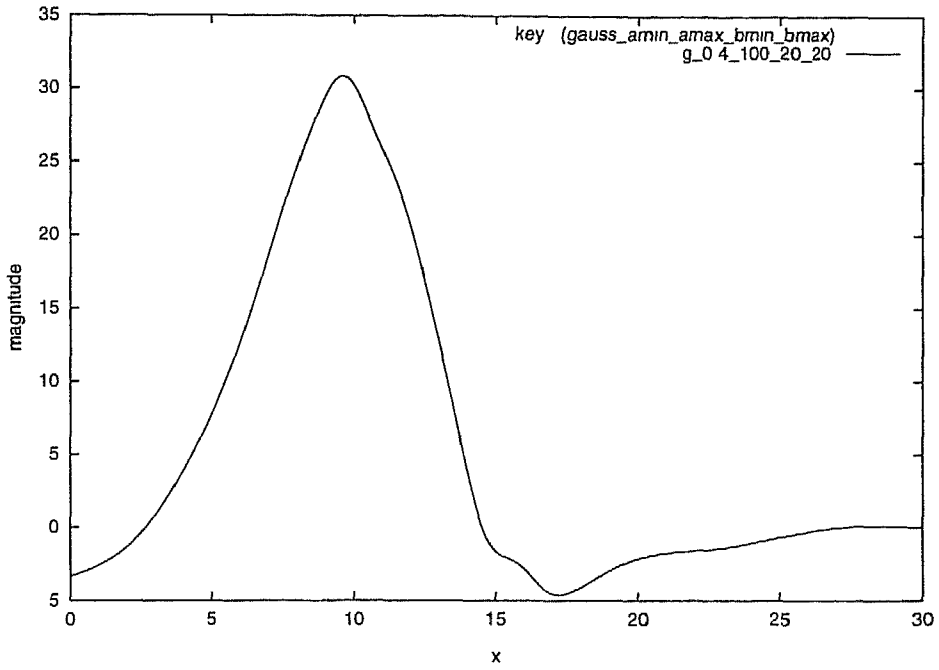


Figure 4 5 Convolution of Ramp and Gate Signal Using Wavelet Transform

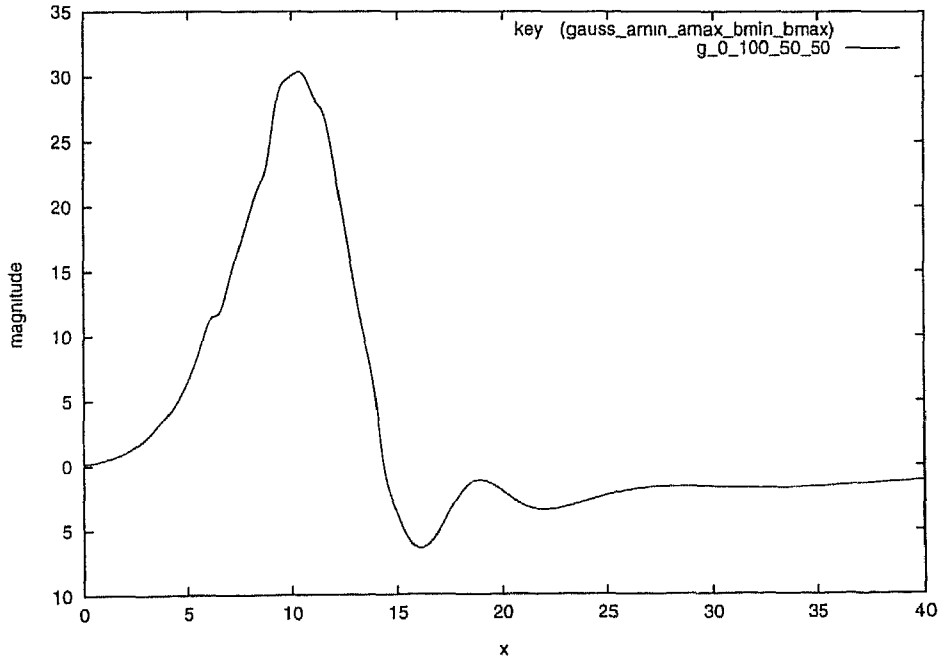


Figure 4 6 Convolution of Ramp and Gate Signal Using Wavelet Transform

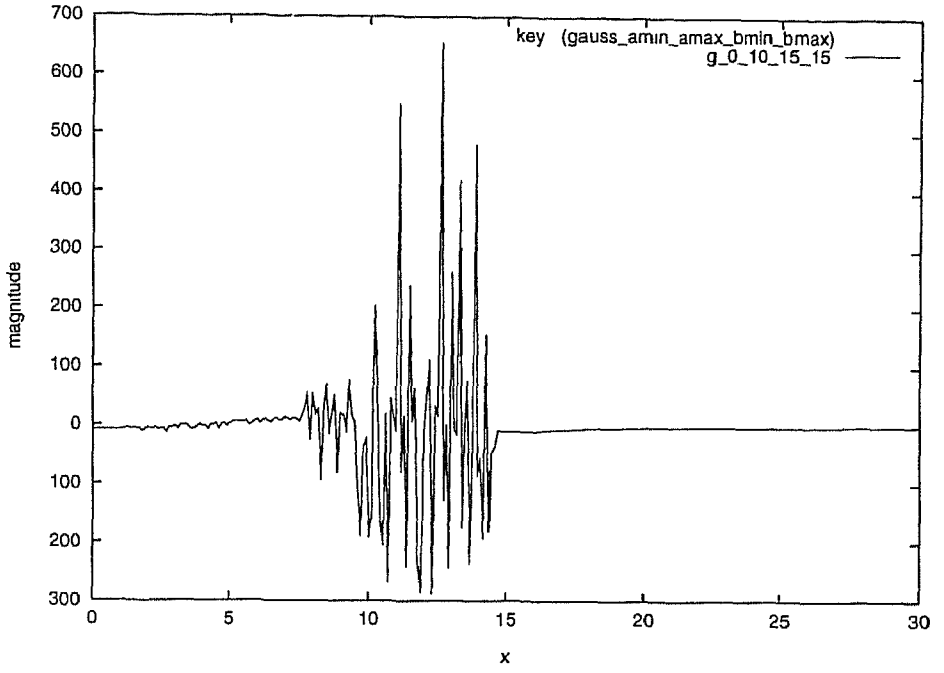


Figure 4 7 Convolution of Ramp and Gate Signal Using Wavelet Transform

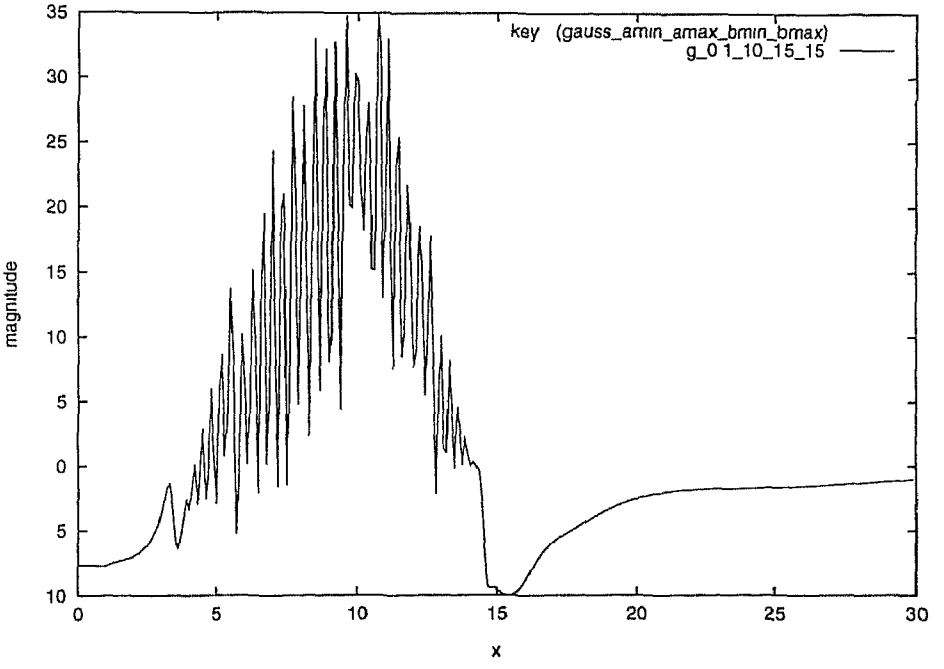


Figure 4 8 Convolution of Ramp and Gate Signal Using Wavelet Transform

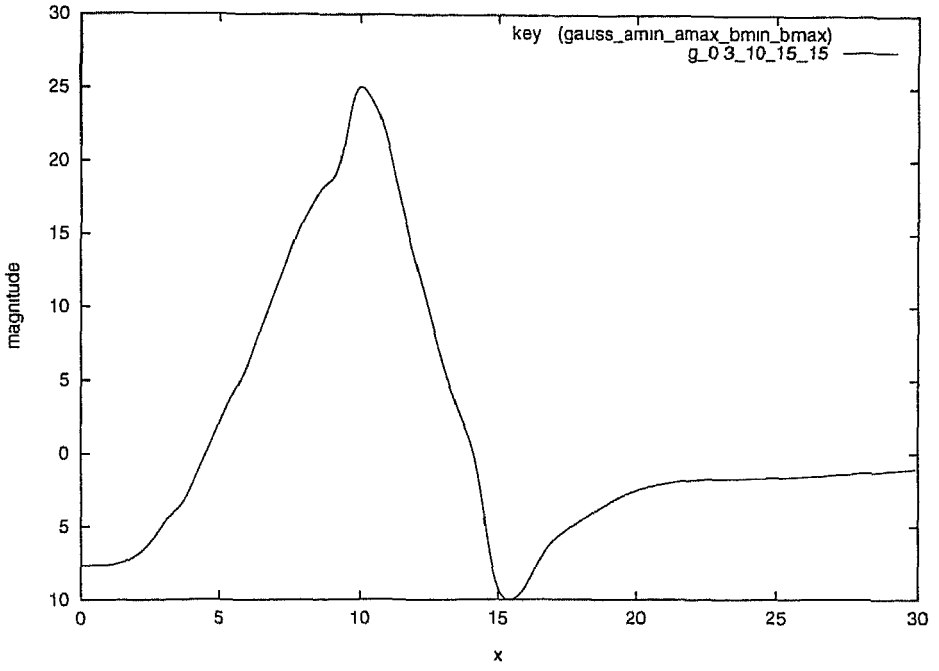


Figure 4 9 Convolution of Ramp and Gate Signal Using Wavelet Transform

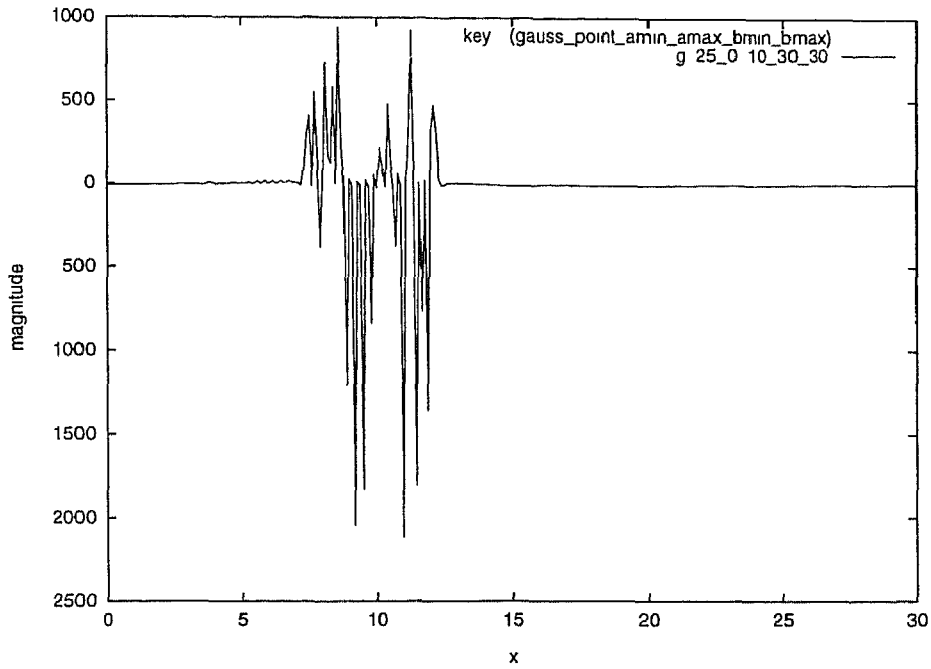


Figure 4 10 Convolution of Ramp and Gate Signal Using Wavelet Transform

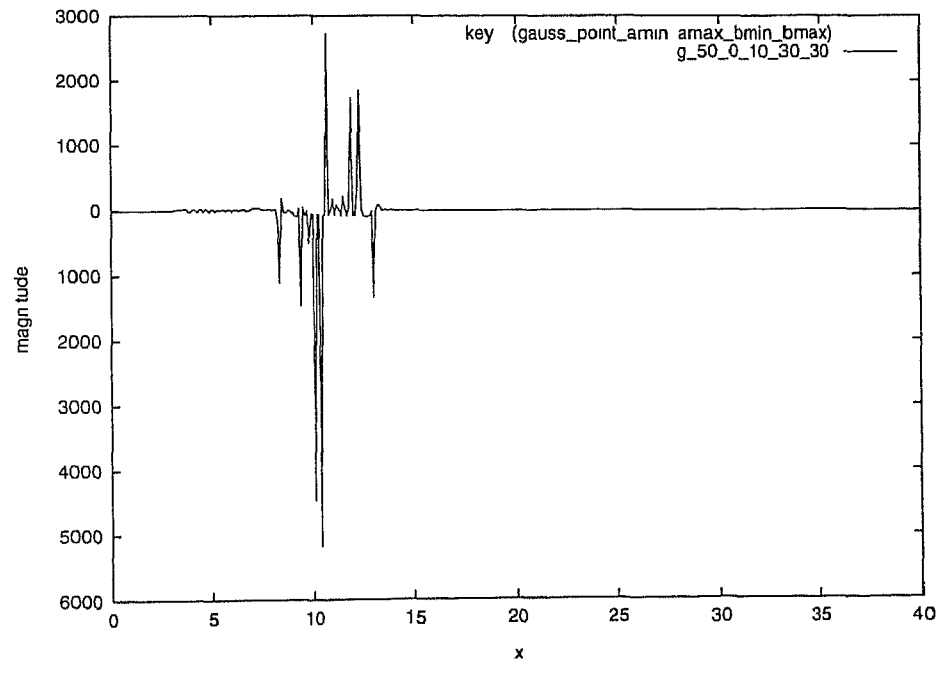


Figure 4 11 Convolution of Ramp and Gate Signal Using Wavelet Transform

The time domain convolution of the exponential signals given by equation 4 14 and equation 4 15 has been calculated and the expression for the convolved output $h(x)$ is given by

$$h(x) = \frac{exp(-\alpha x) - exp(-\beta x)}{(\beta - \alpha)} u(x) \quad if\ \alpha \neq \beta$$
$$= x\, exp(-\alpha x) u(x) \quad if\ \alpha = \beta \qquad (4\ 17)$$

The direct convolution of these signals obtained with the above equation is shown in figure 4 12, and figure 4 13

4 3 Results And Discussion

We have calculated the direct convolution integral of a ramp signal and a gate signal graphically and is given by the equation 4 13 Figure 4 3 shows this as a function of the variable x The direct convolution of the two exponentials, with different decay constants α and β as given in equation 4 14 and equation 4 15 has also been calculated, and the closed form expression for the same is given in equation 4 17 The figure 4 12 and figure 4 13 shows the convolved output for different value of β

The convolution integral so calculated for both sets of input signals has been compared with that obtained using continuous wavelet transform(equation 4 12 and equation 4 16) The comparison is given below

1 The parameters to be chosen for the calculation of the integral given by equation 4 12 are the lower and upper limits of scale a and translation b Those for t and u are known since we have used known signals

2 An inspection of equation 4 12 shows that the integrand is always finite including $a = 0$ and $a = \infty$ for well behaved signals $f(t)$ and $g(u)$ The rise in the integrand at $a = 0$ because of the a^3 term in the denominator is compensated by the exponential term in the numerator, in which a appears in the denominator of the argument (which is negative) of the exponential function Thus in the integration the lower limit of a can be set as zero without any problem Large values of a are unlikely to contribute much because of the a^3 term in the denominator Small

values of a also are unlikely to contribute much because of the structure of the exponential term in the numerator However fixing of the upper cutoff ($a_{\max}$) needs some experimentation These results are discussed later in this section

3 The upper and lower limits of the scale b vary with the value of the scale a We have generated the integrand of the inner integral (integrand over the variable b) in equation 3 16, at different discrete values of a and x and find that the range of b over which the integrand assumes significant values vary with both a and x A range of 50 to +50 would be adequate over the range studied (upto $a = 10$, for different values of x)

4 A severe restriction in our calculation has been the number of points that we use for evaluating the final integral (equation 4 12) The program takes a lot of time if the number of points are increased beyond 20 or at best 30 Thus increasing the range of b to 50 to +50 is not necessarily more accurate than a smaller range of b, because as we increase the range we consider a more complete range of values of the integrand, but since we do not take proportionally large number of points, discretization error enters

5 Figure 4 4 to figure 4 6 shows some of the better results we have obtained Figure 4 4 shows a graph with limits for scale a from 0 to 100 and for the translation b from -20 to +20 It shows a maximum value of approximately 35 at $x \approx 10$ very close to that in figure 4 3 There is a little bit of ripple and the graph starts a little below zero at $x = 0$ but the overall agreement with figure 4 3 is very good Figure 4 5 is also in good agreement with figure 4 3 , but the agreement is not as good as it is between figure 4 3 and figure 4 4 The value of $a_{\mathrm{m\,n}}$ in figure 4 5 is 0 4 and not zero as in figure 4 4 The small yet noticeable deterioration in agreement shows that the contribution of $a = 0$ to 0 4 is not to be ignored Figure 4 6 shows a graph that has same parameters as figure 4 4 but the range of b is from 50 to +50 The agreement with figure 4 3 is not as good as it is with figure 4 4 A wing appears on the right and the maximum amplitude is

depressed The result shows that if number of points for integration does not increase proportionally discretization error increases as range increases The deterioration however is not drastic Increasing the range of b to 100 to +100 and decreasing the integration points to 10 from 20 deteriorates the agreement We note that inclusion of the lower limit of the scale a from $a_{min} = 0\ 0$ to $0\ 4$ improves the result The fact that even with $a_{min} = 0\ 4$ we get very satisfactory result agrees with the conclusion drawn earlier that lower values of a do not contribute significantly We had anticipated that an upper limit of the scale a $a_{max} = 10$ would be adequate because of the factor a^3 in the denominator of equation 4 12 But our results indicate that $a_{max} = 100$ is necessary to obtain good fit

6 If we insist on a smaller $a_{max}(viz\ 10)$ in order to avoid discretization error we obtain a highly oscillatory function $h(x)$ with a maximum at around $x = 10$ but the oscillations mask the actual value of $h(x)$ Such a graph is shown in Fig 4 7 to Fig 4 8 The maximum shoots off to a very large values, much larger than the actual value of 37 5 which is nearly achieved in Fig 4 4 when a_{max} is set at 100

7 We note that the oscillatory nature of calculated $h(x)$ obtained when $a_{min} = 0$ and $a_{max} = 10$ (figure 4 7 and 4 8) disappears when $a_{min} = 0\ 3$ and above (figure 4 9) This combined with the fact that oscillations disappears with $a_{min} = 0$ if $a_{max} = 100$ suggests that the values of a between 0 0 and 0 3 or 0 4 are responsible for the oscillatory behaviour of $h(x)$ In both cases with $a_{min} = 0$ and $a_{max} = 100$ and $a_{min} = 0\ 4$ and $a_{max} = 10$ since the number of points for evaluating the integral is same small values of a are avoided

8 We further note that in Fig 4 8 if we draw a mean curve by averaging the oscillations we get a reasonable approximation to Fig 4 3 The maximum value occurs as it should be at around $x = 10$ but its magnitude is somewhat less than that in Fig 4 3 (~ 30 instead of 37 5) This is not true if $a_{min} = 0$

9 We have also computed $h(x)$ for $a_{max} = 20$ with $a_{min} = 0$ and other small values viz $a_{min} = 0\ 1$ to $0\ 5$ and find highly oscillatory nature in $h(x)$ when $a_{min} = 0\ 1$ or $0\ 2$ but the oscillations disappear at $a_{min} = 0\ 4$ or $0\ 5$

10 The conclusion that smaller values of a are responsible for the oscillatory nature of $h(x)$ is supported by the result that increasing the number of points used in the integration for fixed values of other parameters increase oscillations Fig 4 10 and 4 11 demonstrates this effect

11 Convolution integral of two decaying exponentials the decay constant of the first being 1 and that of the second being 1 10 20 and 50 are calculated by method of continuous wavelet transform with the limits for scale a from 0 to 50 and that for the translation b from -20 to $+20$ The direct convolution using Eq 4 17 has been shown in the Fig 4 12 and 4 13 The convolution using continuous wavelet transform (Eq 4 16) has been shown in the Fig 4 14 to Fig 4 17 The agreement of these plots with Fig 4 12 and 4 13 is quite good Detailed investigations on choice of the limits for the scale a and the translation b are however not done

4 4 Conclusion And Future Work

Convolution integral of two signals can be satisfactorily calculated using the method of continuous wavelet transform with suitable choice of the wavelet function and with appropriate choice of the range of dilation and translation parameters This has been illustrated by evaluating the convolution of a ramp and a gate signal and of two exponentials, with the second derivative of Gaussian wavelet Spurious oscillations in calculating the function $h(x)$ are observed under certain conditions which suggest that small values of the scale parameter a (between 0 to 0 2) cause the oscillatory behaviour The reason for these oscillations are, however, not fully clear Further investigations of this problem for different choice of the range of parameters a and b can be taken up as the future work

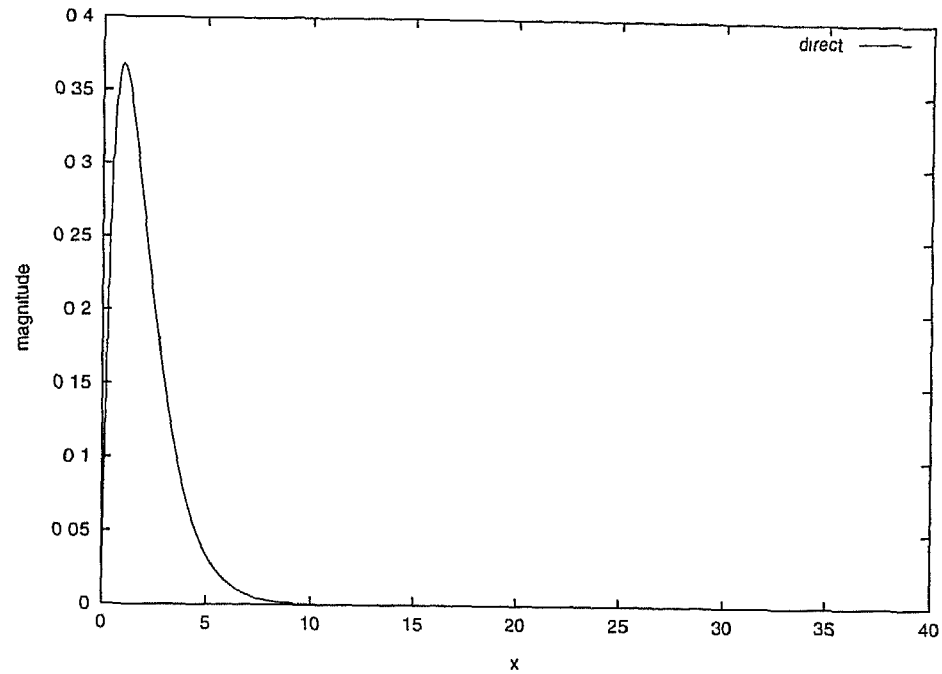


Figure 4 12 Direct Convolution of Two Exponentials ($\alpha = \beta = 1$)

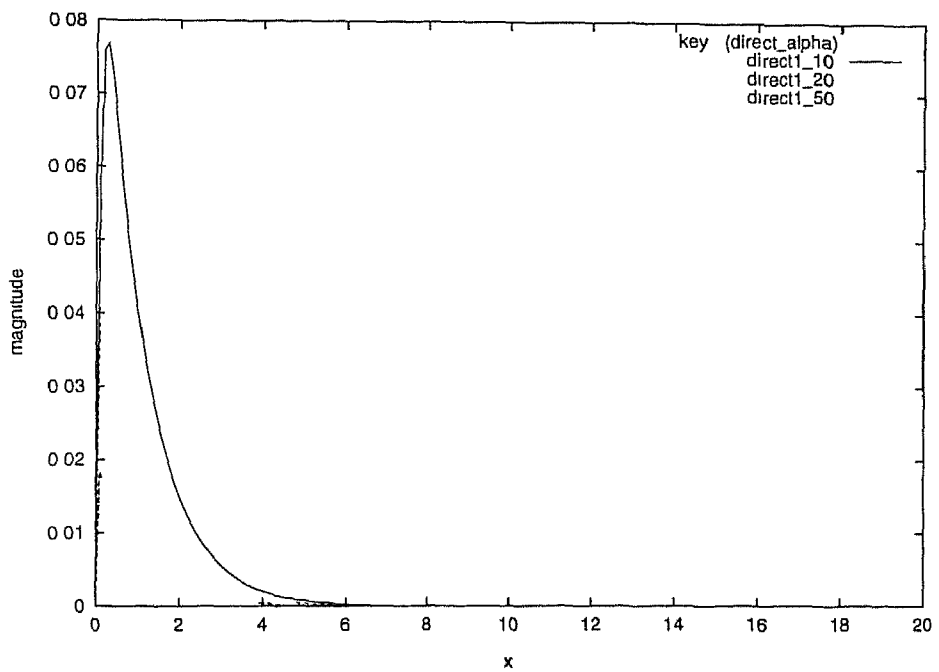


Figure 4 13 Direct Convolution of Two Exponentials with Varying $\alpha(\beta = 1)$

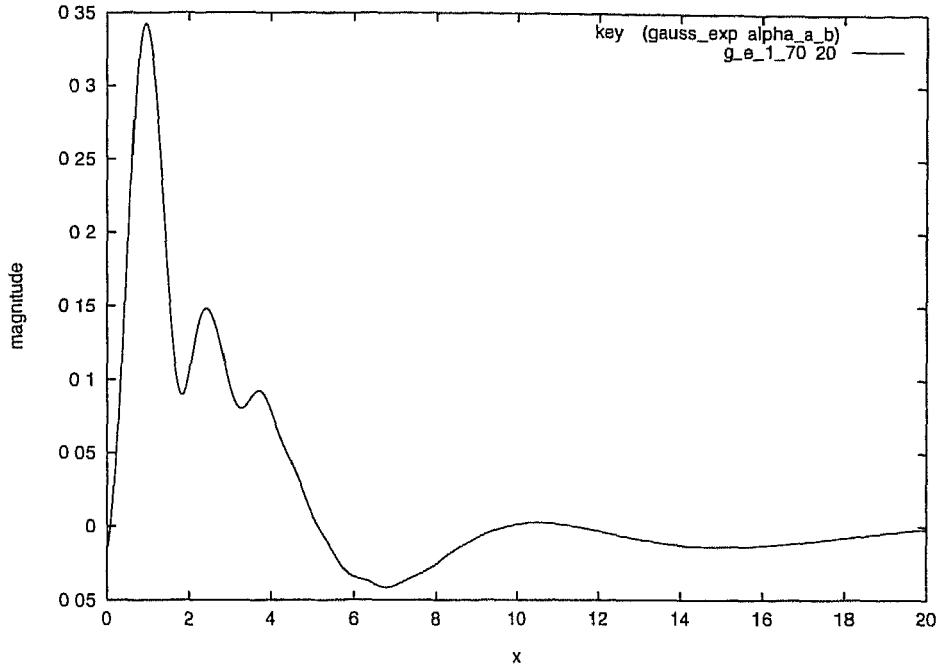


Figure 4 14 Convolution of Two Exponentials with Different $\alpha(\beta = 1)$ Using Wavelet Transform

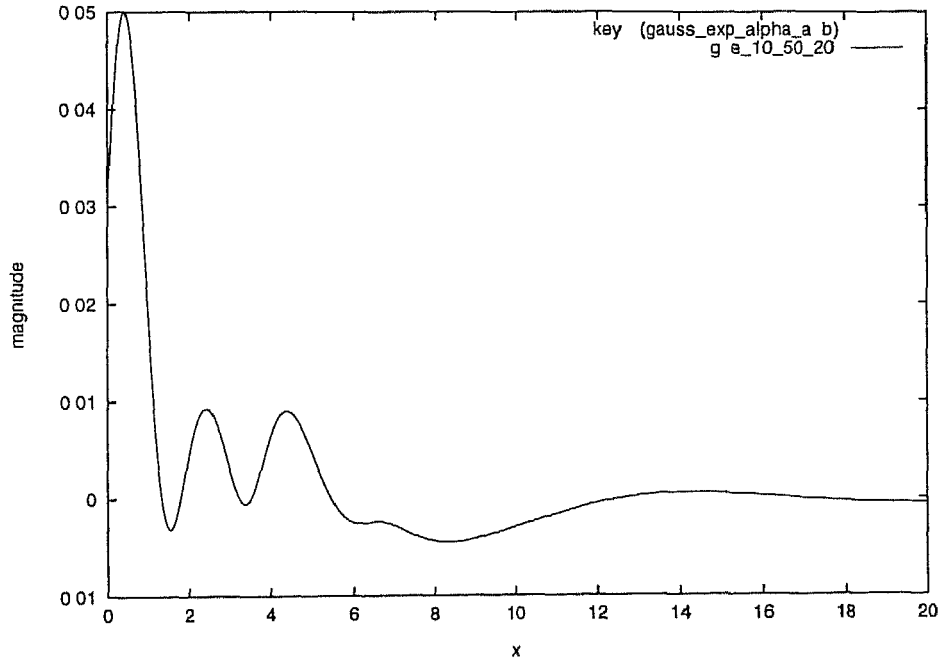


Figure 4 15 Convolution of Two Exponentials with Differnet $\alpha(\beta = 1)$ Using Wavelet Transform

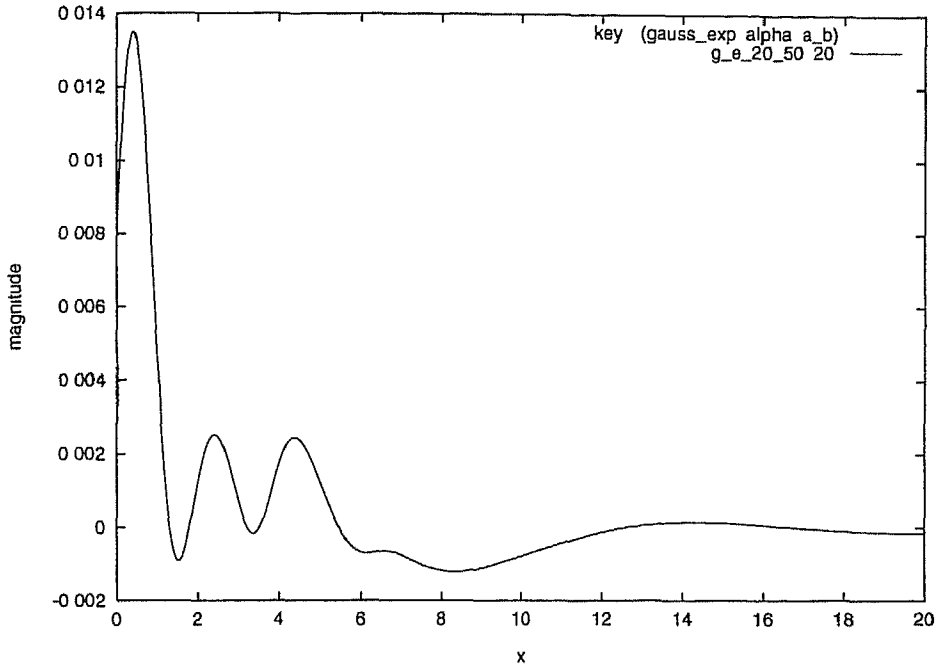


Figure 4 16 Convolution of Two Exponentials with Different $\alpha(\beta = 1)$ Using Wavelet Transform

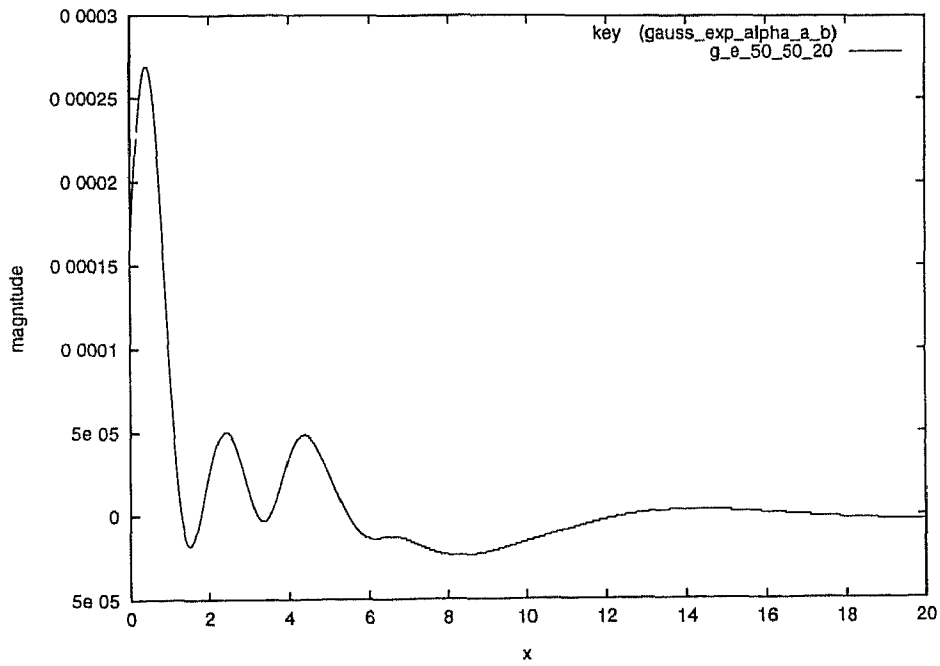


Figure 4 17 Convolution of Two Exponentials with Differnet $\alpha(\beta = 1)$ Using Wavelet Transform

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